Sudden vanishing and reappearance of nonclassical effects: General occurrence of finite-time decays and periodic vanishings of nonclassicality and entanglement witnesses

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Analyses of phenomena exhibiting finite-time decay of quantum entanglement have recently attracted considerable attention. Such decay is often referred to as sudden vanishing (or sudden death) of entanglement, which can be followed by its sudden reappearance (or sudden rebirth). We analyze various finite-time decays (for dissipative systems) and analogous periodic vanishings (for unitary systems) of nonclassical correlations as described by violations of classical inequalities and the corresponding nonclassicality witnesses (or quantumness witnesses), which are not necessarily entanglement witnesses. We show that these sudden vanishings are universal phenomena and can be observed: (i) not only for two- or multi-mode but also for single-mode nonclassical fields, (ii) not solely for dissipative systems, and (iii) at evolution times which are usually different from those of sudden vanishings and reappearances of quantum entanglement.

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I. INTRODUCTION

Decoherence is a crucial obstacle in practical implementations of quantum information processing and quantum state engineering. Quantum entanglement is especially fragile to decoherence. Yu and Eberly [1] (see also earlier studies in Refs. [2]) observed that entanglement decay can occur within a finite time. This effect has been referred to as entanglement "sudden death" or entanglement sudden vanishing (SV) and it can be followed by its sudden reappearance (sudden rebirth—SR) [2–4]. Reference [1] has triggered extensive theoretical research on entanglement loss in various systems (for reviews see Ref. [5]). Entanglement sudden vanishing was also experimentally observed [6–8].

Entanglement SV is often considered to be a new form of decay of quantum entanglement, which presumably was not previously encountered in the dissipation of other physical correlations. Here we would like to point out the *general occurrence of sudden finite-time decays and periodic vanishings of nonclassical correlations*. Namely, the SV and SR effects can also be observed during the evolution of entanglement witnesses [9–11] (for a review see Ref. [12]) and nonclassicality witnesses (also called quantumness witnesses) [13–19] corresponding to violations of classical inequalities.

A standard approach to study the SV and SR of quantum entanglement is based on the analysis of the time evolution of entanglement measures, e.g., the concurrence or, equivalently, the negativity or the relative entropy of entanglement [12]. For a two-qubit system, described by a density matrix $\hat{\rho}$, the concurrence $C(\hat{\rho})$ is defined by [20]:

$$C(\hat{\rho}) = \max\Big(0, 2\max_i \lambda_i - \sum_i \lambda_i\Big), \tag{1}$$

where the λ_i's are the square roots of the eigenvalues of $\hat{\rho}(\hat{\sigma}_2 \otimes \hat{\sigma}_2)\hat{\rho}^*(\hat{\sigma}_2 \otimes \hat{\sigma}_2)$ and $\hat{\sigma}_2$ is the Pauli spin matrix. On the other hand, the negativity can be defined as [9, 21]:

$$N(\hat{\rho}) = \max\Big(0, -2\min_j \mu_j\Big), \tag{2}$$

where μ_j's are the eigenvalues of the partial transpose $\hat{\rho}^{\Gamma}$ and factor 2 is chosen for proper scaling, i.e., to get $N(\hat{\rho}) = 1$ for Bell's states.

It is worth noting that not all the SRs and SVs of entanglement and its witnesses can be considered standard: A SR should appear only after some finite-evolution time after the occurrence of the preceding SV. Specifically, let us now analyze an example: Both $|\cos t|$ and $\max(0, \cos t)$ vanish at $\pi/2$, but only the vanishing of the latter function is associated with the proper SV and SR effects.

Both Eqs. (1) and (2) are given as the maximum of zero and some functions, which clearly explains the occurrence of SVs if $\hat{\rho}$ changes in time. By contrast, SVs do not appear for the modified parameters $C'(\hat{\rho}) = 2\max_i \lambda_i - \sum_i \lambda_i$, and $N'(\hat{\rho}) = -2\min_j \mu_j$, if λ_i and μ_j have continuous derivatives in time.

We deduce that analogous SV and SR effects can be observed for an arbitrary time-dependent parameter $F(t)$, in comparison to some threshold value F_0. From a quantum-mechanical point of view, the most interesting parameters F are the ones which correspond to classical inequalities $F \overset{\rm cl}{\geq} F_0$ that *can* be violated for some *nonclassical* fields, i.e., $F \overset{\rm ncl}{<} F_0$ as indicated by the symbol $\overset{\rm ncl}{<}$. On the other hand, the symbol $\overset{\rm cl}{\geq}$ emphasizes that the corresponding inequality *must* be fulfilled for all *classical* states. Thus, let us truncate such parameter F as follows:

$$F \to \tilde{F} = \max(0, F_0 - F). \tag{3}$$

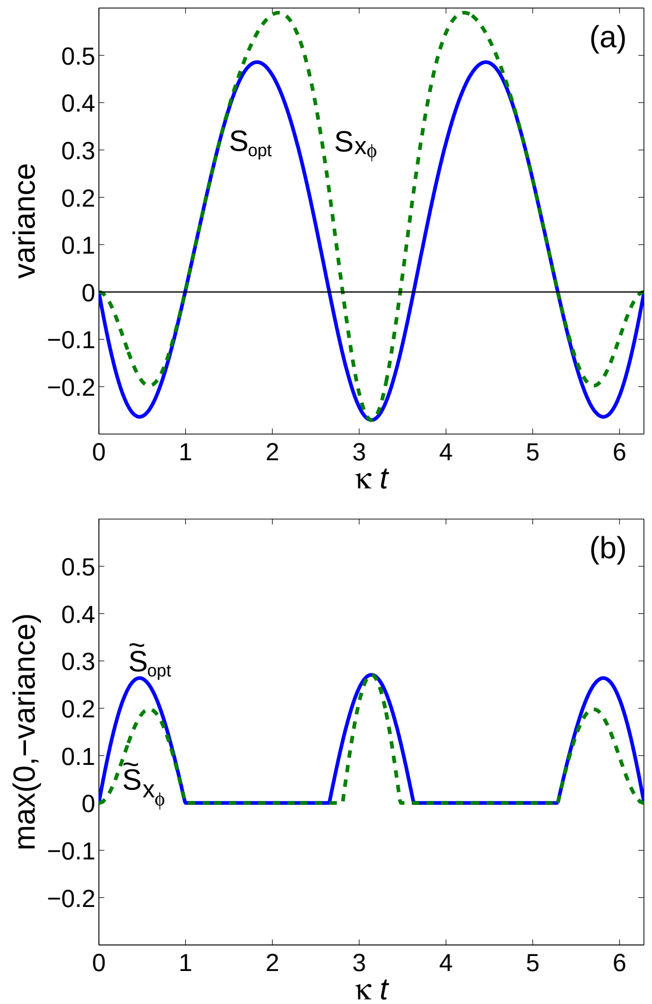

FIG. 1: (Color online) A simple explanation of how to observe the SV and SR of nonclassicality witnesses using, as an example, the unitary evolution of single-mode squeezing in the anharmonic oscillator model given by the Hamiltonian (58): (a) normally-ordered variances S_{x_ϕ} (dashed curve) and $S_{\rm opt}$ (solid curve), given by Eqs. (60) and (61), (b) truncated normally-ordered variances $\tilde{S}_{x_\phi}$ (dashed curve) and $\tilde{S}_{\rm opt}$ (solid curve), given by Eqs. (15) and (19), respectively. Quadrature squeezing occurs if $S_{x_\phi} < 0$ or, equivalently, if the truncated witness $\tilde{S}_{x_\phi} > 0$. Principal squeezing occurs if $S_{\rm opt} < 0$ or if the truncated witness $\tilde{S}_{\rm opt} > 0$. Here, $|\alpha_0|^2 = 1/2$, $\phi_0 = \phi = 0$, and $S_0 = 0$. By including damping, one would observe finite-time decays, analogously to the standard sudden decays of entanglement.

A simple illustration of this concept is shown in Fig. 1. For brevity, such F and $\tilde{F}$ will be referred to as the untruncated and truncated *nonclassicality witnesses*, respectively. The redefinition of the witnesses is a key concept in observing the SV and SR effects.

In the next sections we give general arguments and present some specific examples of phenomena and nonclassicality witnesses to support our conclusions.

In Sec. II we recall a definition of nonclassicality and present a general method of constructing truncated nonclassicality witnesses that can exhibit both the SV and SR effects. In Sec. III, we discuss methods of constructing truncated entanglement witnesses. We also give a few simple examples of truncated nonclassicality and entanglement witnesses. Their evolution in some prototype physical models is studied in Secs. IV-VI. We conclude in Sec. VII.

II. NONCLASSICALITY WITNESSES

In order to test (and characterize) the nonclassical behavior of a given state $\hat\rho$ unambiguously, we use the multimode Cahill-Glauber s-parametrized quasiprobability distribution (QPD) functions defined for $-1 \le s \le 1$ by [22]:

$$\mathcal{W}^{(s)}(\boldsymbol{\alpha}) = \frac{1}{\pi}\,{\rm Tr}\left(\hat\rho\prod_{k=1}^{M}\hat{T}^{(s)}(\alpha_k)\right), \tag{4}$$

where

$$\hat{T}^{(s)}(\alpha_k) = \frac{1}{\pi}\int \exp\left(\alpha_k\xi^* - \alpha_k^*\xi + \frac{s}{2}|\xi|^2\right)\hat{D}(\xi)\,{\rm d}^2\xi, \tag{5}$$

$\hat{D}(\xi)$ is the displacement operator, $\boldsymbol{\alpha}$ is a complex multivariable $(\alpha_1, \alpha_2, ..., \alpha_M)$, and M is the number of modes. In special cases (for $s = 1, 0, -1$), the QPD reduces to the standard Glauber-Sudarshan P function, Wigner W function, and Husimi Q function, respectively.

A well-known criterion of nonclassicality (or quantumness) is based on the P function (see, e.g., Refs. [23]):

Definition 1 *A state $\hat\rho$ is considered* nonclassical *if its Glauber-Sudarshan P function is not a classical probability density (i.e., it is nonpositive). Otherwise the state $\hat\rho$ is called classical.*

We use this definition of nonclassicality although we are aware of its drawbacks (see, e.g., Ref. [24]). It is also worth noting that this definition is often extended by a requirement of nonsingularity. That is, a classical P function cannot be more singular than Dirac's δ function. But, in fact, the singularity of the P function is implied by its nonpositivity (see, e.g., Ref. [19]).

Definition 1 can be equivalently formulated via a complete set of nonclassicality witnesses corresponding to violations of classical inequalities. Here we apply the method of constructing nonclassicality witnesses proposed in Refs. [13, 14] and developed in Refs. [19, 25]. Alternatively, one can apply an approach used by Alicki *et al.* [15–17].

Let us analyze an arbitrary M-mode operator $\hat{f} \equiv \hat{f}(\hat{\mathbf{a}}, \hat{\mathbf{a}}^\dagger)$ as a function of the annihilation, $\hat{\mathbf{a}} \equiv (\hat{a}_1, \hat{a}_2, ..., \hat{a}_M)$, and creation, $\hat{\mathbf{a}}^\dagger$, operators. The P function enables a direct calculation of the normally-ordered (denoted by ::) expectation values of the Hermitian operator $\hat{f}^\dagger\hat{f}$ as follows:

$$\langle: \hat{f}^\dagger\hat{f} :\rangle \;=\; \int {\rm d}^2\boldsymbol{\alpha}\;|f(\boldsymbol{\alpha},\boldsymbol{\alpha}^*)|^2 P(\boldsymbol{\alpha},\boldsymbol{\alpha}^*). \tag{6}$$

Then one can apply another criterion of nonclassicality [13, 14]:

Criterion 1 *A state $\hat{\rho}$ is classical if $\langle : \hat{f}^\dagger\hat{f} : \rangle \geq 0$ for all functions $\hat{f}$. Conversely, if $\langle : \hat{f}^\dagger\hat{f} : \rangle < 0$ for some $\hat{f}$ then the state $\hat{\rho}$ is nonclassical.*

These conditions can be compactly written as $\langle : \hat{f}^\dagger\hat{f} : \rangle \overset{\rm cl}{\geq} 0$ and $\langle : \hat{f}^\dagger\hat{f} : \rangle \overset{\rm ncl}{<} 0$. By analogy with definitions of entanglement witness (see the following section), the normally-ordered Hermitian operator $: \hat{f}^\dagger\hat{f} :$ can be referred to as (nonlinear) nonclassicality (or quantumness) witness [14]. For convenience, we call the nonclassicality witness (and also entanglement witness) not only an observable but also its expectation value. Note that the understanding of nonclassicality witnesses is not strictly limited to operators (see, e.g., Refs. [17, 18]).

By writing $\hat{f} = \sum_i^N c_i\hat{f}_i$, where c_i are arbitrary complex numbers, one obtains

$$\langle : \hat{f}^\dagger\hat{f} : \rangle \ = \ \sum_{i,j} c_i^* c_j \langle : \hat{f}_i{}^\dagger \hat{f}_j : \rangle. \tag{7}$$

The normally-ordered moments $\langle : \hat{f}_i{}^\dagger \hat{f}_j : \rangle$ can be grouped into the following matrix:

$$M_{\hat{f}}^{(\rm n)}(\hat{\rho}) = \begin{pmatrix} \langle : \hat{f}_1^\dagger\hat{f}_1 : \rangle & \langle : \hat{f}_1^\dagger\hat{f}_2 : \rangle & \cdots & \langle : \hat{f}_1^\dagger\hat{f}_N : \rangle \\ \langle : \hat{f}_2^\dagger\hat{f}_1 : \rangle & \langle : \hat{f}_2^\dagger\hat{f}_2 : \rangle & \cdots & \langle : \hat{f}_2^\dagger\hat{f}_N : \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle : \hat{f}_N^\dagger\hat{f}_1 : \rangle & \langle : \hat{f}_N^\dagger\hat{f}_2 : \rangle & \cdots & \langle : \hat{f}_N^\dagger\hat{f}_N : \rangle \end{pmatrix}. \tag{8}$$

We call (nonlinear) nonclassicality witnesses not only $: \hat{f}^\dagger\hat{f} :$ and $\langle : \hat{f}^\dagger\hat{f} : \rangle$ but also the matrices of normally-ordered moments $M_{\hat{f}}^{(\rm n)}(\hat{\rho})$ and their functions (e.g., determinants). The importance of this approach is motivated by the following nonclassicality criterion [13, 19]:

Criterion 2 *A state $\hat{\rho}$ is nonclassical if there exists $\hat{f}$, such that $\det[M_{\hat{f}}^{(\rm n)}(\hat{\rho})]$ is negative.*

Thus, if these nonclassicality witnesses are truncated according to Eq. (3), one can predict infinitely many different kinds of SV and SR effects. Note that a given nonclassicality witness reveals only some specific and limited properties of nonclassical states.

It is worth stressing that nonclassicality witnesses are (usually) not measures of nonclassicality. A question arises whether SV and SR effects can also be observed for some nonclassicality measures. Below we give an example of quantum dynamics leading to the SV and SR of nonclassicality witnesses but *not* of nonclassicality measures.

A. Examples of truncated nonclassicality witnesses

To find nontrivial examples of SV and SR of some nonclassicality witnesses, which are not necessarily entanglement witnesses (studied in the following section), we analyze the squeezing (or sub-Poisson statistics) of the photon-number difference $(\hat{n}_1 - \hat{n}_2)$ in two systems. This squeezing occurs if the normally-ordered variance

$$S = \langle : [\Delta(\hat{n}_1 - \hat{n}_2)]^2 : \rangle \tag{9}$$

is negative, where $\Delta\hat{O} \equiv \hat{O} - \langle\hat{O}\rangle$, with $\hat{O} = \hat{n}_1 - \hat{n}_2$. It is a purely nonclassical effect as $S \overset{\rm cl}{\geq} 0$ holds for any classical fields. Note that $S + S_0 \overset{\rm cl}{\geq} 0$ also holds for any classical fields, where $S_0 \geq 0$ is a threshold value which can be chosen to be arbitrary. Thus, one can analyze a kind of "strong" squeezing if $S + S_0 \overset{\rm ncl}{<} 0$. In order to observe the SV and SR of this strong squeezing we truncate the squeezing parameter S as follows:

$$\tilde{S} = \max\left(0, -\langle : [\Delta(\hat{n}_1 - \hat{n}_2)]^2 : \rangle - S_0\right) \overset{\rm ncl}{>} 0. \tag{10}$$

By replacing $\Delta(\hat{n}_1 - \hat{n}_2)$ by $(\hat{n}_1 - \hat{n}_2)$ in Eq. (9), one can consider another normally-ordered witness $\tilde{D}'$ resulting from the classical inequality

$$D' = \langle : (c_1\hat{n}_1 + c_2\hat{n}_2 + c_3)^2 : \rangle + |c_4|^2 \overset{\rm cl}{\geq} 0 \tag{11}$$

assuming real parameters c_k ($k = 1, 2, 3, 4$). In the following, we apply

$$\tilde{D} = \max(0, -\langle : (\hat{n}_1 - \hat{n}_2 + D_0)^2 : \rangle) \overset{\rm ncl}{>} 0, \tag{12}$$

which is a special case of $\tilde{D}'$ for $(c_1, c_2, c_3, c_4) = (1, -1, D_0, 0)$.

So far we have only analyzed two-mode witnesses. Clearly, it is also possible to observe the SV and SR during the time evolution of multi-mode but also single-mode witnesses of nonclassicality. We give only two examples of photon-number and quadrature squeezings:

(i) Single-mode photon-number squeezing (also called sub-Poisson photon-number statistics) occurs if Mandel's Q-parameter is negative, i.e., $\langle : (\Delta\hat{n})^2 : \rangle / \langle : \hat{n} : \rangle \overset{\rm ncl}{<} 0$. This nonclassical effect can also be described by the truncated witness

$$\tilde{Q} = \max\left(0, -\frac{\langle : (\Delta\hat{n})^2 : \rangle}{\langle : \hat{n} : \rangle}\right) \overset{\rm ncl}{>} 0. \tag{13}$$

(ii) The standard ($S_0 = 0$) and strong ($S_0 > 0$) M-mode quadrature squeezing can be defined by

$$S_{x_\phi} = \langle : (\Delta\hat{x}_{\boldsymbol{\phi}})^2 : \rangle \overset{\rm ncl}{<} (-S_0), \tag{14}$$

or, equivalently, via the truncated squeezing witness

$$\tilde{S}_{x_\phi} = \max(0, -\langle : (\Delta\hat{x}_{\boldsymbol{\phi}})^2 : \rangle - S_0) \overset{\rm ncl}{>} 0, \tag{15}$$

where $\boldsymbol{\phi} = (\phi_1, \phi_2, ..., \phi_M)$. The multimode quadrature operator is given by [23]:

$$\hat{x}_{\boldsymbol{\phi}} = \sum_{m=1}^{M} c_m\, \hat{x}_m(\phi_m) \tag{16}$$

is a sum of single-mode phase-rotated quadratures

$$\hat{x}_m(\phi_m) = \hat{a}_m \exp(i\phi_m) + \hat{a}_m^\dagger \exp(-i\phi_m). \quad (17)$$

The truncated nonclassicality witness $\tilde{S}_{x_\phi}$, given by Eq. (15), can also be used in a single-mode case. The $\boldsymbol{\phi}$-optimized quadrature squeezing is referred to as *principal* squeezing and is defined by the witness [19, 26]:

$$S_{\rm opt} = \min_{\boldsymbol{\phi}} S_{x_\phi} \overset{\rm ncl}{<} 0, \quad (18)$$

or the truncated witness

$$\tilde{S}_{\rm opt} = \max(0, -S_{\rm opt} - S_0) = \max_{\boldsymbol{\phi}} \tilde{S}_{x_\phi} \overset{\rm ncl}{>} 0. \quad (19)$$

Note that all entanglement witnesses are also nonclassicality witnesses, but not vice versa. An example of the single-mode evolution exhibiting the SV and SR of the nonclassicality witnesses, corresponding to the quadrature and principal squeezing, is shown in Fig. 1 for the anharmonic model described in Sec. VI.

Explicit examples of many other two- and multimode nonclassicality witnesses, corresponding to violations of classical inequalities, can be found in, e.g., Refs. [14, 19, 23, 27–30].

III. ENTANGLEMENT WITNESSES

An effective method of constructing entanglement witnesses can be based on the Shchukin-Vogel entanglement criterion [31] (or its generalizations [32]) for distinguishing states with positive partial transposition from those with nonpositive partial transposition (NPT).

In analogy to the matrices of normally-ordered moments $M^{(\rm n)}_{\hat{f}}(\hat{\rho})$, given by Eq. (8), one can define the following matrix of partially-transposed moments:

$$M_{\hat{f}}(\hat{\rho}^\Gamma) = \begin{pmatrix} \langle \hat{f}_1^\dagger \hat{f}_1\rangle^\Gamma & \langle \hat{f}_1^\dagger \hat{f}_2\rangle^\Gamma & \cdots & \langle \hat{f}_1^\dagger \hat{f}_N\rangle^\Gamma \\ \langle \hat{f}_2^\dagger \hat{f}_1\rangle^\Gamma & \langle \hat{f}_2^\dagger \hat{f}_2\rangle^\Gamma & \cdots & \langle \hat{f}_2^\dagger \hat{f}_N\rangle^\Gamma \\ \vdots & \vdots & \ddots & \vdots \\ \langle \hat{f}_N^\dagger \hat{f}_1\rangle^\Gamma & \langle \hat{f}_N^\dagger \hat{f}_2\rangle^\Gamma & \cdots & \langle \hat{f}_N^\dagger \hat{f}_N\rangle^\Gamma \end{pmatrix}, \quad (20)$$

where $\hat{f} = \sum_i^N c_i \hat{f}_i$ for arbitrary complex numbers c_i, $\langle \hat{f}_i^\dagger \hat{f}_j\rangle^\Gamma \equiv {\rm tr}(\hat{f}_i^\dagger \hat{f}_j \hat{\rho}^\Gamma)$ and Γ denotes partial transposition. The Shchukin-Vogel entanglement criterion [31, 32] can be written as:

Criterion 3 *A bipartite state $\hat{\rho}$ is NPT if and only if there exists $\hat{f}$, such that ${\rm det}[M_{\hat{f}}(\hat{\rho}^\Gamma)]$ is negative.*

This criterion resembles Criterion 2 of the nonclassicality. Thus, analogously to the nonclassicality witnesses, we refer to such matrices $M_{\hat{f}}(\hat{\rho}^\Gamma)$ of partially transposed moments and their functions (like determinants) as (state-dependent nonlinear) *entanglement witnesses*. It is worth noting that according to the original definition, entanglement witnesses correspond to observables rather than expectation values [9]: An entanglement witness is a Hermitian operator $\hat{W}$ such that ${\rm tr}(\hat{W}\hat{\rho}_{\rm sep}) \geq 0$ for all separable states $\hat{\rho}_{\rm sep}$, while ${\rm tr}(\hat{W}\hat{\rho}_{\rm ent}) < 0$ for some entangled states $\hat{\rho}_{\rm ent}$. This concept was later generalized to nonlinear entanglement witnesses [10, 11]. Although our usage of the term entanglement witness differs slightly from the original usage, we believe that it can improve readability of our paper, while keeping unchanged the main idea of entanglement witnesses.

Here we give only two examples of such entanglement witnesses based on Criterion 3. Let us apply the following Hillery-Zubairy classical inequalities [33]:

$$\langle \hat{n}_1\hat{n}_2\rangle \overset{\rm cl}{\geq} |\langle \hat{a}_1\hat{a}_2^\dagger\rangle|^2, \quad \langle \hat{n}_1\rangle\langle \hat{n}_2\rangle \overset{\rm cl}{\geq} |\langle \hat{a}_1\hat{a}_2\rangle|^2, \quad (21)$$

where $\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i$ is the photon number operator, and $\hat{a}_i$ ($\hat{a}_i^\dagger$) is the annihilation (creation) operator for mode $i = 1, 2$. Thus, we can define the following truncated witnesses

$$\tilde{H} = \max(0, |\langle \hat{a}_1\hat{a}_2^\dagger\rangle|^2 - \langle \hat{n}_1\hat{n}_2\rangle) \overset{\rm ent}{>} 0, \quad (22)$$

$$\tilde{H}' = \max(0, |\langle \hat{a}_1\hat{a}_2\rangle|^2 - \langle \hat{n}_1\rangle\langle \hat{n}_2\rangle) \overset{\rm ent}{>} 0. \quad (23)$$

which can be positive only for some *entangled* states, as marked by the symbol $\overset{\rm ent}{>}$. These inequalities can be derived in various ways, e.g., from the Cauchy-Schwarz inequality [33] or from entanglement criteria based on partial transposition [31, 32]. Thus, $\tilde{H}$ and $\tilde{H}'$ are entanglement witnesses, so the SV of the concurrence implies also the SV of $\tilde{H}(t)$ and $\tilde{H}'(t)$ (if they were nonzero for some evolution times). It is worth noting that the inequalities in Eq. (21) are satisfied not only by separable states but also by all classical states (marked by $\overset{\rm cl}{\geq}$) since they can be derived from nonclassicality criteria based on the P function [19].

Another simple choice of an entanglement witness can be related, e.g., to the violation of Bell's inequality. For two-qubit states, a degree of violation of Bell's inequality, in its version due to Clauser, Horne, Shimony, and Holt (CHSH) [34], can be defined as [35, 36]:

$$B^2(\hat{\rho}) \equiv \max\left[0, \max_{j<k}(u_j + u_k) - 1\right], \quad (24)$$

where u_j ($j = 1, 2, 3$) are the eigenvalues of $U_{\hat{\rho}} = T_{\hat{\rho}}^T\, T_{\hat{\rho}}$, $T_{\hat{\rho}}$ is a real matrix with elements $t_{ij} = {\rm Tr}\left[\hat{\rho}\,(\hat{\sigma}_i \otimes \hat{\sigma}_j)\right]$, and $\hat{\sigma}_j$ are Pauli's spin matrices. For brevity, although not precisely, B is often referred to as a *nonlocality* (measure). Analogously to the concurrence C, the nonlocality B is defined as the maximum of zero and another quantity, which implies that it is possible to observe the SV and SR of $B(t)$ in a dynamical scenario.

If a two-qubit state $\hat{\rho}$ violates Bell's inequality then it is also entangled, but not vice versa, i.e., there are mixed states $\hat{\rho}$ (e.g., Werner's states discussed below), for which $C(\hat{\rho}) > 0$ and $B(\hat{\rho}) = 0$. Thus, $B(\hat{\rho})$ can be considered as an *entanglement witness*. The SV of an entanglement measure implies the SV of an nonlocality measure (if the latter was nonzero at some evolution time). Note that for two-qubit pure states $B(\hat{\rho}) = C(\hat{\rho})$, so in this case the nonlocality is not only an entanglement witness but also an entanglement measure.

IV. SUDDEN DECAYS OF NONCLASSICALITY WITNESSES FOR NONINTERACTING MODES

Let us first give a simple example of the environment-induced sudden vanishing of the entanglement that is closely related to the original idea of finite-time sudden decays. As a generalization, we also study sudden vanishings of several other nonclassicality witnesses, which occur at times different than those for the entanglement vanishing.

By contrast to the following sections, we analyze the entanglement of two modes (qubits), which are not directly interacting with each other but only with independent reservoirs. Specifically, we describe the SV of the nonclassicality of initially entangled states, due to interaction with the reservoirs under Markov's approximation, by applying the standard master equation for the reduced density operator $\hat{\rho}$:

$$\frac{\partial}{\partial t}\hat{\rho} = \sum_{k=1,2}\frac{\gamma_k}{2}[\bar{n}_k(2\hat{a}_k^\dagger\hat{\rho}_k\hat{a}_k - \hat{a}_k\hat{a}_k^\dagger\hat{\rho} - \hat{\rho}\hat{a_k}\hat{a}_k^\dagger) + (\bar{n}_k+1)(2\hat{a}_k\hat{\rho}\hat{a}_k^\dagger - \hat{a}_k^\dagger\hat{a}_k\hat{\rho} - \hat{\rho}\hat{a}_k^\dagger\hat{a}_k)] - \frac{i}{\hbar}[\hat{\mathcal{H}}_S,\hat{\rho}], \tag{25}$$

where γ_k are the damping rates, $\bar{n}_k$ are the mean thermal photon numbers, $\bar{n}_k = \{\exp[\hbar\omega_k/(k_B T)] - 1\}^{-1}$, T is the reservoirs temperature at thermal equilibrium, and k_B is Boltzmann's constant. We assume the reservoirs to be at zero temperature, so we set $\bar{n}_1 = \bar{n}_2 = 0$. The Hamiltonian $\hat{\mathcal{H}}_S$ is just the sum of free Hamiltonians for the two noninteracting system modes. We solve the master equation by applying the Monte Carlo wave function simulation with the collapse operators $\hat{c}_{1k} = \sqrt{\gamma(1+\bar{n}_k)}\hat{a}_k$ and $\hat{c}_{2k} = \sqrt{\gamma\bar{n}_k}\hat{a}_k^\dagger$ [37].

It is worth noting that from the standard physical point of view, the quantum entanglement between two systems, and the related violation of Bell's inequalities, can be considered if the systems are spatially separated and are physically uncoupled [38]. It is seen that this model (contrary to the models studied in the following section) satisfies the second condition.

Our example of the environment-induced sudden vanishing of quantumness and nonlocality is provided for a system coupled to two independent reservoirs. It is worth mentioning that common reservoirs in some cases can also enhance entanglement both for two qubits and two modes. This is possible due to a mixing mechanism rather than an induced interaction among them [39].

Let us analyze the decoherence of the initial Werner-like state defined as [36]:

$$\hat{\rho}_m(0) = p|\Psi_m\rangle\langle\Psi_m| + \frac{1-p}{4}\hat{I}, \tag{26}$$

for $0 \le p \le 1$, $m = 1$, and $|\Psi_1\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. Here, $\hat{I}$ is the identity operator. Under this initial condition, the solution of the master equation can be given in the standard computational basis as [36]:

$$\hat{\rho}_1(t) = \frac{1}{4}\begin{bmatrix} h^{(+)} & 0 & 0 & 2p\sqrt{g_1g_2} \\ 0 & h_1^{(+)} & 0 & 0 \\ 0 & 0 & h_2^{(+)} & 0 \\ 2p\sqrt{g_1g_2} & 0 & 0 & (1+p)g_1g_2 \end{bmatrix}, \tag{27}$$

where $h^{(+)} = (2-g_1)(2-g_2)+pg_1g_2$, $h_k^{(+)} = g_{3-k}[2-(1+p)g_k]$, and $g_k = \exp(-\gamma_k t)$ for $k = 1,2$. The concurrence and nonlocality decay as follows [36]:

$$C(t) = \max\left\{0, \frac{1}{2}\sqrt{g_1g_2}\Big(2p - \sqrt{[2-(1+p)g_1][2-(1+p)g_2]}\Big)\right\}, \tag{28}$$

$$B^2(t) = \max\left(0, 2p^2g_1g_2 - 1\right), \tag{29}$$

respectively. For comparison, we also calculate the decays of the two witnesses of the photon-number-difference correlations:

$$\tilde{S}(t) = \max\left[0, \frac{1}{4}(g_1^2 + g_2^2 + 2pg_1g_2) - S_0\right], \tag{30}$$

$$\tilde{D}(t) = \max\left[0, \frac{1}{2}g_1g_2(1+p) - D_0^2 - D_0(g_1-g_2)\right]. \tag{31}$$

For simplicity, let us assume now the same reservoir damping rate γ, so $g_1 = g_2 \equiv g$. Then, the SV times for the above entanglement and nonclassicality witnesses can be different from each other as they are given by

$$t_{\rm SV}^{(C)} = \frac{1}{\gamma}\ln\left(\frac{1+p}{2(1-p)}\right), \tag{32}$$

$$t_{\rm SV}^{(B)} = \frac{1}{\gamma}\ln\left(\sqrt{2}p\right), \tag{33}$$

$$t_{\rm SV}^{(\tilde{S})} = \frac{1}{2\gamma}\ln\left(\frac{1+p}{2S_0}\right), \tag{34}$$

$$t_{\rm SV}^{(\tilde{D})} = \frac{1}{2\gamma}\ln\left(\frac{1+p}{2D_0^2}\right). \tag{35}$$

The results are shown in Fig. 2 assuming some specific values of the damping constant γ and the initial Werner state $\hat{\rho}_1(0)$ with parameter p.

In conclusion, we have given a simple example of the decaying entanglement between two qubits, which are not directly interacting with each other, but they are only coupled to the environment. We have observed the SVs of the two nonclassicality witnesses, which are different from the SVs of the entanglement and nonlocality measures.

V. PERIODIC SUDDEN VANISHING OF NONCLASSICALITY WITNESSES OF INTERACTING MODES

A. Frequency conversion model

Here we give an illustrative example of *periodic* sudden vanishing of nonclassicality witnesses during a unitary evolution of two interacting modes. This is in contrast to the

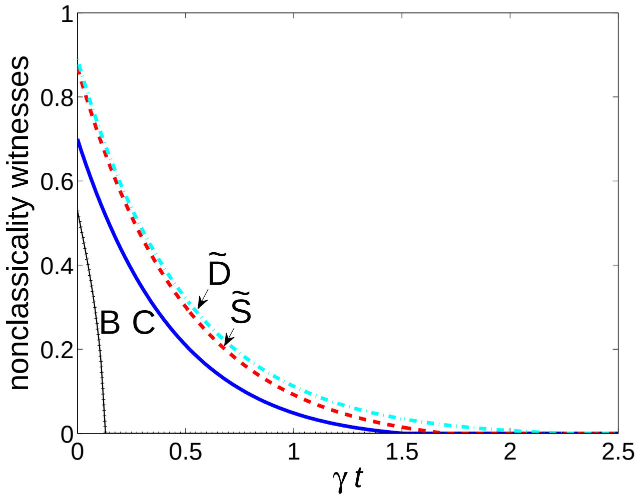

FIG. 2: (Color online) An example of the environment-induced sudden vanishing of the nonclassicality witnesses for two *noninteracting* modes. The damping model is described in Sec. IV for the initial Werner-like state ρ_1 with $p = 0.8$. Key: the concurrence C (solid curve), nonlocality B (dotted curve), and two witnesses describing the photon-number-difference correlations: $\tilde{S}$ (dashed curve) for $S_0 = 0.03$ and $\tilde{D}$ (dot-dashed curve) for $D_0 = 0.1$.

standard analysis of sudden decays applied solely to dissipative systems. Note that one can easily include the dissipation (as studied, e.g., in the former section) to observe the proper finite-time sudden decays and SRs analogous to the standard ones.

As a simple model to study SV and SR, let us study the parametric frequency conversion described by the interaction Hamiltonian

$$\hat{\mathcal{H}} = \hbar\kappa[\hat{a}_1^\dagger \hat{a}_2 \exp(-i\Delta\omega t) + \hat{a}_1 \hat{a}_2^\dagger \exp(i\Delta\omega t)], \tag{36}$$

which is a prototype Hamiltonian describing two linearly-coupled harmonic oscillators. It can be applied to a variety of physical phenomena including the process of exchanging photons between two optical fields of different frequencies: a signal mode with frequency ω_1 and an idler mode with frequency ω_2. Then $\hat{a}_1$ and $\hat{a}_2$ are the annihilation operators for the signal and idler modes, respectively, and κ is the real coupling constant. For simplicity, we assume a resonant case $\Delta\omega = \omega + \omega_2 - \omega_1$.

The well-known solutions of the Heisenberg equations of motion for the signal, $\hat{b}_1(t)$, and idler, $\hat{b}_2(t)$, modes are given by [40]:

$$\begin{aligned} \hat{b}_1(t) &= \hat{a}_1 \cos(\kappa t) - i\,\hat{a}_2 \sin(\kappa t), \\ \hat{b}_2(t) &= \hat{a}_2 \cos(\kappa t) - i\hat{a}_1 \sin(\kappa t). \end{aligned} \tag{37}$$

The corresponding solution of the Schrödinger equation is

$$|\psi(t)\rangle = \sum_{n_1,n_2} c_{n_1,n_2} \frac{[\hat{b}_1^\dagger(-t)]^{n_1}}{\sqrt{n_1!}} \frac{[\hat{b}_2^\dagger(-t)]^{n_2}}{\sqrt{n_2!}} |00\rangle \tag{38}$$

assuming that the system is initially in a superposition of Fock states, $|\psi(0)\rangle = \sum_{n_1,n_2} c_{n_1,n_2}|n_1,n_2\rangle$. The total number of photons is a constant of motion, $\hat{n}_1(t) + \hat{n}_2(t) =$const.

An important property of the (undamped) parametric frequency model is that the nonclassicality of an arbitrary state is unchanged during its evolution. By applying the results of Refs. [41–43], one can find that the time evolution of the QPD for the frequency-converter model, described by Eq. (36), with arbitrary initial fields is simply given by

$$\mathcal{W}^{(s)}(\alpha_1,\alpha_2,t) = \mathcal{W}^{(s)}\left[\beta_1(\alpha_1,\alpha_2,-t), \beta_2(\alpha_1,\alpha_2,-t), 0\right], \tag{39}$$

where $\beta_{1,2}(\alpha_1,\alpha_2,t)$ are the solutions of the corresponding *classical* equations of motion for the frequency conversion model:

$$\begin{aligned} \beta_1(\alpha_1,\alpha_2,t) &= \alpha_1 \cos(\kappa t) - i\alpha_2 \sin(\kappa t), \\ \beta_2(\alpha_1,\alpha_2,t) &= \alpha_2 \cos(\kappa t) - i\alpha_1 \sin(\kappa t). \end{aligned} \tag{40}$$

Equation (39) means that the two-mode QPD for the model discussed is constant along classical trajectories. Thus, if the initial fields are nonclassical, their degree of nonclassicality (as defined, e.g., in Refs. [44–46]) remains unchanged at any evolution times of the system. But yet we can observe SV and SR of entanglement and nonclassicality witnesses as will be shown in the following subsections.

B. Evolution of a pure state

Let us first analyze the parametric frequency conversion for the initial state $|\psi(0)\rangle = |01\rangle$. The system evolves, according to Eq. (38), into

$$|\psi(t)\rangle = \cos(\kappa t)|01\rangle - i\sin(\kappa t)|10\rangle. \tag{41}$$

It is a nonclassical state described by the following *singular* (so negative) P function

$$P(\alpha_1,\alpha_2,t) = \delta[\beta_1(\alpha_1,\alpha_2,t)] \left(1 + \frac{\partial}{\partial\beta_2(\alpha_1,\alpha_2,t)} \times \frac{\partial}{\partial\beta_2^*(\alpha_1,\alpha_2,t)}\right) \delta[\beta_2(\alpha_1,\alpha_2,t)], \tag{42}$$

which is given in terms of Dirac's δ function, its derivative, and the solutions of the classical equations of motion, given by Eq. (40). Elementary calculations lead to the following expressions for the concurrence and nonlocality

$$C(t) = B(t) = |\sin(2\kappa t)|, \tag{43}$$

the entanglement witness describing the violation of the first Hillery-Zubairy inequality

$$\tilde{H}(t) = \frac{1}{4}\sin^2(2\kappa t), \tag{44}$$

and the nonclassicality witnesses for the photon-number-difference correlations

$$\begin{aligned} \tilde{S}(t) &= \max\left[0, \cos^2(2\kappa t) - S_0\right], & (45) \\ \tilde{D}(t) &= \max\left\{0, D_0[2\cos(2\kappa t) - D_0]\right\}. & (46) \end{aligned}$$

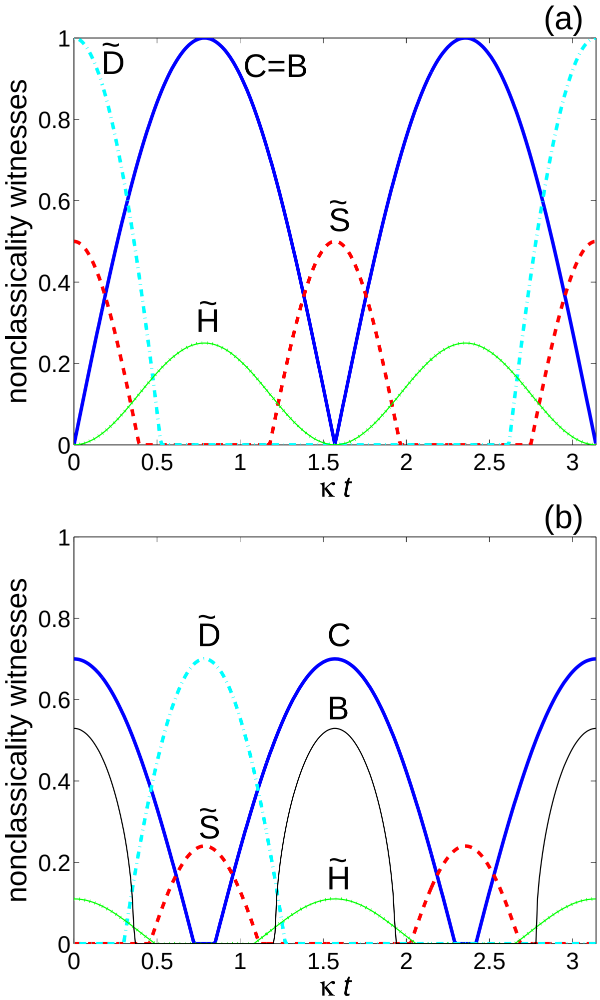

FIG. 3: (Color online) Simple examples of the sudden vanishing and reappearance of the concurrence and other truncated nonclassicality witnesses for two interacting modes. The unitary evolution of the frequency model is shown assuming: (a) the initial pure state $|01\rangle$ discussed in Sec. V B and (b) the initial mixed state, given by Eq. (47) with $p = 0.8$, analyzed in Sec. V C. Key: C (thick solid curve) is the concurrence, B (thin solid curve) is the nonlocality; $\tilde{H}$ (dotted curve) is the entanglement witness, given by Eq. (22), describing the violation of the first Hillery-Zubairy inequality; $\tilde{S}$ (dashed curve) for $S_0 = 1/2$ and $\tilde{D}$ (dot-dashed curve) for $D_0 = 1$ are nonclassicality witnesses describing the photon-number-difference correlations, which are given by Eqs. (10) and (12), respectively. From the standard point of view, a SR should appear only after some finite evolution time after the occurrence of the preceding SV. It is seen that this condition is satisfied for all the witnesses of the mixed-state evolution (b), but only for some witnesses of the pure-state evolution (a).

Analogously, one also finds the photon-number sub-Poisson statistics of the fields as described by the modified Mandel parameters $\tilde{Q}_1 = \sin^2(\kappa t)$ and $\tilde{Q}_2 = \cos^2(\kappa t)$. All these nonclassical witnesses exhibit periodic SV and SR effects as shown in Fig. 3(a). For example, the SV and SR of the concurrence corresponds to the maximum value of $\tilde{S}$. Analogously, we could observe the out-of-phase SVs and SRs of Mandel's Q parameters, which can be clearly understood by recalling the classical-like interpretation of two linearly coupled oscillators when one of them is initially excited ($Q_2 > 0$) and the other is unexcited ($Q_1 = 0$). During the evolution, the excitation is transferred periodically between the oscillators.

One can raise an objection concerning the above example that a SV of the concurrence is instantly followed by a SR, so they are not the proper SV and SR effects. The same behavior is found for the other witnesses including $\tilde{D}$ for $D_0 = 0$, and $\tilde{S}$ for $S_0 = 0$. From the more standard, or more orthodox, point of view, a SV (of some witness) should not be instantly followed by a SR. By contrast, the SV times differ from the SR times for $\tilde{D}$ with $D_0 > 0$ and for $\tilde{S}$ with $S_0 > 0$ [as shown in Fig. 3(a)] that is required in the orthodox approach.

Other, even more convincing, examples of the SV and SR effects can be found by analyzing the evolution of initially mixed states as will be shown below.

C. Evolution of a mixed state

Let us choose the initial state to be a Werner-like state $\hat{\rho}_0(0)$, given by Eq. (26) for $m = 0$ and $|\Psi_0\rangle = (|01\rangle - i|10\rangle)/\sqrt{2}$. This state evolves as follows

$$\hat{\rho}_0(t) = p|\Psi_0(t)\rangle\langle\Psi_0(t)| + \frac{1-p}{4}\hat{I}, \tag{47}$$

where

$$|\Psi_0(t)\rangle = \frac{1}{\sqrt{2}}\left[f_-(t)|01\rangle - if_+(t)|10\rangle\right] \tag{48}$$

with $f_\pm(t) = \cos(\kappa t) \pm \sin(\kappa t)$. We find the following evolutions of the entanglement witnesses and the corresponding times of the first SV:

$$C(t) = \max[0, p|c| - (1-p)/2] \;\Rightarrow\; t_{\rm SV}^{(C)} = f\left(\frac{1-p}{2p}\right), \tag{49}$$

$$B^2(t) = \max[0, p^2\left(1+c^2\right) - 1] \;\Rightarrow\; t_{\rm SV}^{(B)} = f\left(\frac{\sqrt{1-p^2}}{p}\right), \tag{50}$$

$$\tilde{H}(t) = \frac{1}{4}\max[0, (pc)^2 - (1-p)] \;\Rightarrow\; t_{\rm SV}^{(\tilde{H})} = f\left(\frac{\sqrt{1-p}}{p}\right), \tag{51}$$

where $f(x) = \arccos x/(2\kappa)$ and $c = \cos(2\kappa t)$. The first SR occurs at the time

$$\kappa t_{\rm SR}^{(i)} = \pi/2 - \kappa t_{\rm SV}^{(i)} \tag{52}$$

for $i = C, B, \tilde{H}$. It is seen in Fig. 3(b) for $p = 0.8$ that the first SVs and SRs occur in the following order:

$$t_{\rm SV}^{(B)} < t_{\rm SV}^{(\tilde{H})} < t_{\rm SV}^{(C)} \;\Rightarrow\; t_{SR}^{(B)} > t_{\rm SR}^{(\tilde{H})} > t_{\rm SR}^{(C)}. \tag{53}$$

On the other hand, the nonclassicality witnesses $\tilde{D}$ and $\tilde{S}$, given by Eqs. (10) and (12), respectively, evolve as

$$\tilde{S}(t) = \max[0, (1-p)/2 + p^2\sin^2(2\kappa t) - S_0], \tag{54}$$

$$\tilde{D}(t) = \max[0, (1-p)/2 + 2D_0 p\sin(2\kappa t) - D_0^2]. \tag{55}$$

For $S_0 = 0$ and $p < 1$, we do not observe a complete vanishing of $\tilde{S}(t)$. For $S_0 = 0$ and $p = 1$ (which corresponds to the initial Bell state), $\tilde{S}(t)$ periodically vanishes to zero and instantly increases, so it is not a good example of the SV and SR effects. However, for $0 < p < 1$ we can observe the proper SV and SR effects as shown in Fig. 3(b). The first SVs occur at the times

$$t_{\rm SV}^{(\tilde{S})} = \frac{\pi}{4\kappa} + f\left(\frac{\sqrt{2S_0+p-1}}{\sqrt{2}p}\right), \quad (56)$$

$$t_{\rm SV}^{(\tilde{D})} = \frac{\pi}{4\kappa} + f\left(\frac{2D_0^2+p-1}{4D_0 p}\right), \quad (57)$$

and the first SRs occur at $t_{\rm SR}^{(\tilde{S})} = \pi/\kappa - t_{\rm SV}^{(\tilde{S})}$ and $t_{\rm SR}^{(\tilde{D})} = 3\pi/(2\kappa) - t_{\rm SV}^{(\tilde{D})}$. Note that the first appearances of these witnesses occur at earlier times, i.e., $t = \pi/(2\kappa) - t_{\rm SV}^{(i)}$ for $i = \tilde{S}, \tilde{D}$. It is seen that we can always choose threshold values S_0 and D_0 for any $0 < p < 1$ in such a way to observe the SVs and SRs of these witnesses for the photon-number-difference correlations at arbitrary evolution times also when the system is disentangled.

VI. PERIODIC SUDDEN VANISHING OF NONCLASSICALITY WITNESSES FOR A SINGLE MODE

Finally, let us analyze a single-mode anharmonic oscillator described by the interaction Hamiltonian

$$\hat{\mathcal{H}} = \frac{1}{2}\hbar\kappa(\hat{a}^\dagger)^2\hat{a}^2. \quad (58)$$

This is a prototype model of various fundamental phenomena including the optical Kerr effect. For simplicity, here we refer to this effect only. Under this interaction, the initial coherent state $|\alpha_0\rangle$ evolves periodically into a nonclassical state

$$|\psi(t)\rangle = e^{-|\alpha_0|^2/2}\sum_{n=0}^{\infty}\frac{\alpha_0^n}{\sqrt{n!}}\exp\left[\frac{i}{2}n(n-1)\tau\right]|n\rangle, \quad (59)$$

where τ is a rescaled time κt. It is worth noting that the Kerr state, given by Eq. (59), becomes at some evolution times a superposition of macroscopically distinguishable two [47] or more [48] coherent states, which are often referred to as the Schrödinger cat and kitten states, respectively. Among many nonclassical intriguing properties of the model (see, e.g., Ref. [49] and references therein), the Kerr state exhibits high-degree quadrature squeezing [26, 50]. We find that the single-mode normally-ordered variance S_{x_ϕ} of the quadrature operator $\hat{x}_\phi = \hat{x}_1(\phi) = \hat{a}\exp(-i\phi) + \hat{a}^\dagger\exp(i\phi)$ can be compactly written as follows:

$$S_{x_\phi} = 2|\alpha_0|^2[1 + f_{12}\cos(\tau_{12}+\tau) - f_{21}(\cos\tau_{21}+1)] \quad (60)$$

in terms of the auxiliary functions defined by $\tau_{kl} = k|\alpha_0|^2\sin(l\tau)+2(\phi-\phi_0)$ and $f_{kl} = \exp\{k|\alpha_0|^2[\cos(l\tau)-1]\}$ with $\alpha_0 = |\alpha_0|\exp(i\phi_0)$. Quadrature squeezing occurs if $S_{x_\phi} \overset{\rm ncl}{<} 0$ or, equivalently, if the truncated witness $\tilde{S}_{x_\phi} \overset{\rm ncl}{>} 0$, defined by Eq. (15) with Eq. (60) and $\boldsymbol{\phi} = \phi$. For simplicity, we set a threshold value S_0 to be zero in this section and in Fig. 1. By applying the results of Refs. [26, 50], we can compactly write the ϕ-optimized variance $S_{\rm opt}$ describing the principal squeezing as follows

$$S_{\rm opt}(t) = 2|\alpha_0|^2\Big(1 - f_{21} - \sqrt{f_{22}+f_{41}-2f_{12}f_{21}\cos\tau'}\Big), \quad (61)$$

where $\tau' = \tau_{12} - \tau_{21} + \tau$. Analogously to the former squeezing criteria, the principal squeezing occurs if $S_{\rm opt} \overset{\rm ncl}{<} 0$ or if the truncated witness $\tilde{S}_{\rm opt} \overset{\rm ncl}{>} 0$, as given by Eqs. (19) and (61). Our results are presented in Fig. 1 for some specific amplitude of the initial coherent state.

Note that the periodic vanishing of the entanglement and nonclassicality witnesses, analyzed here and in Sec. V, should not be confused with the oscillations of the entanglement measures in systems interacting with non-Markovian reservoirs (see, e.g., Ref. [51]). The SV and SR effects in such systems have different character than studied here. Mazzola *et al.* [51] observed the oscillations in short times, which disappear after some finite time and are related to the non-Markovian character of the reservoirs. In contrast, in the examples presented here, the periodic behavior of the nonclassicality witnesses persists as being related to the unitary evolution of the states.

It is worth stressing again that the aperiodic SV and SR effects, which are analogous to the typical sudden decays of the entanglement, can be observed by inclusion of the dissipation. Assuming Markov's approximation, one can apply the master equation, given by Eq. (25) in a special case for a single mode ($k = 1$). Then the SVs and SRs become aperiodic and the final SV occurs after some evolution time, which depends on the dissipation. However, the dissipation is not a necessary condition for the SV occurrence in this model.

The SV and SR of the entanglement in two-mode dissipative coupled Kerr models was studied in Ref. [52]. Here we showed that the periodic SV and SR of squeezing can be observed even in the single-mode nondissipative Kerr model. This example confirms our conclusion of the general occurrence of the SV and SR of nonclassicality witnesses even for single-mode undamped systems.

VII. CONCLUSIONS

We have applied the concepts of the SV and SR of quantum entanglement measures to study the SV and SR of entanglement and nonclassicality witnesses.

Our main observations can be summarized as follows:

(i) SVs can be encountered not only in the dissipation of entanglement but also of other nonclassical correlation parameters, related to violations of classical inequalities [19, 23].

(ii) SVs occur not only in the dissipation of bipartite or multipartite (multimode) interacting or noninteracting systems but also in a single-qubit or single-mode systems. Our examples include single-mode squeezing of photon number, squeezing of quadrature operators [23], and violations of other classical inequalities [19].

(iii) Non-dissipative systems, which are initially even in pure states, can also exhibit periodic SVs of nonclassical phenomena and the related nonclassicality witnesses. For instance, the quadrature squeezing of light in a Kerr medium exhibits periodic SVs for some finite periods of time. In order to observe the proper finite-time sudden decays analogous to the standard sudden decays of entanglement [1], one should add dissipation by coupling such systems to the environment. The damping causes irregularity and loss of periodicity of the evolution of the nonclassicality witnesses. We can conclude that the damping accelerates the occurrence of the first SVs but it is not a necessary condition for their occurrence.

With the help of the nonclassicality criteria [19, 25] and entanglement criteria [31, 32], based on moments of the annihilation and creation operators, as discussed in Secs. II and III, it is possible to construct infinitely many nonclassicality and entanglement witnesses. These witnesses, after truncation according to Eq. (3), can exhibit the SV and SR effects when analyzing their time evolution.

We hope that these observations might motivate deeper analysis of SV and SR of various nonclassicality witnesses in specific models and also in experimental scenarios.

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[1] T. Yu and J. H. Eberly, Phys. Rev. Lett. **93**, 140404 (2004).

[2] K. Życzkowski, P. Horodecki, M. Horodecki, and R. Horodecki, Phys. Rev. A **65**, 012101 (2001); A. K. Rajagopal and R.W. Rendell, Phys. Rev. A **63**, 022116 (2001); L. Jakóbczyk and A. Jamróz, Phys. Lett. A **318**, 318 (2003); S. Daffer, K. Wódkiewicz, and J. K. McIver, Phys. Rev. A **67**, 062312 (2003); A. Miranowicz, J. Phys. A **37**, 7909 (2004).

[3] R. Tanaś and Z. Ficek, J. Opt. B: Quantum Semiclass. Opt. **6**, S90 (2004); Z. Ficek and R. Tanaś, Phys. Rev. A **74**, 024304 (2006).

[4] C.E. López, G. Romero, F. Lastra, E. Solano, and J.C. Retamal, Phys. Rev. Lett. **101**, 080503 (2008).

[5] T. Yu and J.H. Eberly, Science **323**, 598 (2009); J.H. Eberly, T. Yu, Science **316**, 555 (2007).

[6] M.P. Almeida, F. de Melo, M. Hor-Meyll, A. Salles, S.P. Walborn, P.H. Souto Ribeiro, and L. Davidovich, Science **316**, 579 (2007).

[7] J. Laurat, K.S. Choi, H. Deng, C.W. Chou, and H.J. Kimble, Phys. Rev. Lett. **99**, 180504 (2007).

[8] J.S. Xu, C.F. Li, M. Gong, X.B. Zou, C.H. Shi, G. Chen, and G.C. Guo, Phys. Rev. Lett. **104**, 100502 (2010).

[9] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).

[10] P. Hyllus and J. Eisert, New J. Phys. **8**, 51 (2006).

[11] O. Gühne and N. Lütkenhaus, Phys. Rev. Lett. **96**, 170502 (2006).

[12] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).

[13] E.V. Shchukin, Th. Richter, and W. Vogel, Phys. Rev. A **71**, 011802(R) (2005).

[14] J.K. Korbicz, J.I. Cirac, J. Wehr, and M. Lewenstein, Phys. Rev. Lett. **94**, 153601 (2005).

[15] R. Alicki and N. Van Ryn, J. Phys. A **41**, 062001 (2008); R. Alicki, M. Piani and N. Van Ryn, J. Phys. A **41**, 495303 (2008).

[16] M. Żukowski, Phys. Rev. A **79**, 024103 (2009).

[17] S.N. Filippov and V.I. Man'ko, Phys. Scr. **79**, 055007 (2009).

[18] A.A. Semenov, D.Y. Vasylyev, and B.I. Lev, J. Phys. B **39**, 905 (2006).

[19] A. Miranowicz, M. Bartkowiak, X. Wang, Y.X. Liu, and F. Nori, Phys. Rev. A **82**, 013824 (2010).

[20] W.K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).

[21] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).

[22] K.E. Cahill and R.J. Glauber, Phys. Rev. **177**, 1857 (1969).

[23] W. Vogel and D.G. Welsch, *Quantum Optics* (Wiley-VCH, Weinheim, 2006).

[24] A. Wünsche, J. Opt. B: Quantum Semiclass. Opt. **6**, 159 (2004).

[25] W. Vogel, Phys. Rev. Lett. **100**, 013605 (2008).

[26] A. Lukš, V. Peřinová, and Z. Hradil, Acta Phys. Polon. **A74**, 713 (1988).

[27] M.D. Reid and D.F. Walls, Phys. Rev. A **34**, 1260 (1986).

[28] J. Peřina, *Quantum Statistics of Linear and Nonlinear Optical Phenomena* (Reidel, Dortrecht, 1991).

[29] X. Wang, A. Miranowicz, Y.X. Liu, C.P. Sun, and F. Nori, Phys. Rev. A **81**, 022106 (2010); X. Wang, J. Ma, L. Song, X. Zhang, and X. Wang, Phys. Rev. E **82**, 056205 (2010).

[30] A. Miranowicz, J. Bajer, H. Matsueda, M.R.B. Wahiddin, and R. Tanaś, J. Opt. B: Quantum Semiclass. Opt. **1**, 511 (1999); A. Miranowicz, H. Matsueda, J. Bajer, M.R.B. Wahiddin, and R. Tanaś, *ibid.* **1**, 603 (1999).

[31] E.V. Shchukin and W. Vogel, Phys. Rev. Lett. **95**, 230502 (2005); A. Miranowicz and M. Piani, Phys. Rev. Lett. **97**, 058901 (2006).

[32] A. Miranowicz, M. Piani, P. Horodecki, and R. Horodecki, Phys. Rev. A **80**, 052303 (2009).

[33] M. Hillery and M.S. Zubairy, Phys. Rev. Lett. **96**, 050503 (2006).

[34] J.F. Clauser, M.A. Horne, A. Shimony, and R.A. Holt, Phys. Rev. Lett. **23**, 880 (1969).

[35] R. Horodecki, P. Horodecki, and M. Horodecki, Phys. Lett. A **200**, 340 (1995).

[36] A. Miranowicz, Phys. Lett. A **327**, 272 (2004).

[37] S.M. Tan, J. Opt. B: Quantum Semiclass. Opt. **1**, 424 (1999).

[38] A. Wójcik, Science **301**, 1183 (2003).

[39] F. Benatti, R. Floreanini, and M. Piani, Phys. Rev. Lett. **91**,

070402 (2003); F. Benatti and R. Floreanini, J. Phys. A **39**, 2689 (2006).
[40] W. Louisell, *Radiation and Noise in Quantum Electronics* (McGraw-Hill, New York, 1964).
[41] R. J. Glauber, Phys. Lett. **21**, 650 (1966).
[42] L. Mišta, Czech. J. Phys. B **19**, 443 (1969).
[43] A. Miranowicz, H. Matsueda, M.R.B. Wahiddin, and J. Bajer, Acta Phys. Slov. **48**, 293 (1998).
[44] C.T. Lee, Phys. Rev. A **44**, R2775 (1991).
[45] N. Lütkenhaus and S.M. Barnett, Phys. Rev. A **51**, 3340 (1995).
[46] A. Kenfack and K. Życzkowski, J. Opt. B: Quantum Semiclass. Opt. **6**, 396 (2004).
[47] B. Yurke and D. Stoler, Phys. Rev. Lett. **57**, 13 (1986).
[48] A. Miranowicz, R. Tanaś, and S. Kielich, Quantum Opt. **2**, 253 (1990); R. Tanaś, Ts. Gantsog, A. Miranowicz, and S. Kielich, J. Opt. Soc. Am. B **8**, 1576 (1991).
[49] R. Tanaś, in: *Theory of Non-Classical States of Light*, eds. V. Dodonov and V.I. Man'ko (Taylor and Francis, London, 2003) p. 267-312.
[50] R. Tanaś, A. Miranowicz, and S. Kielich, Phys. Rev. A **43**, 4014 (1991).
[51] L. Mazzola, S. Maniscalco, J. Piilo, K.A. Suominen, and B.M. Garraway, Phys. Rev. A **79**, 042302 (2009).
[52] A. Miranowicz and W. Leoński, J. Phys. B **39**, 1683 (2006); A. Kowalewska-Kudaszyk and W. Leoński, J. Opt. Soc. Am. B **26**, 1289 (2009); J. Phys. B **43**, 205503 (2010).